Lecture 27

Signed measures.

Let $(X, \mathcal{M})$ be measurable space.

Def. 1 A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ that at most takes one of the values $\pm\infty$ s.t.

(i) $\nu(\emptyset) = 0$

(ii) If $E = \bigcup_{k=1}^{\infty} E_k$ is a disjoint union of $E_k \in \mathcal{M}$, then $\nu(E) = \sum_{k=1}^{\infty} \nu(E_k)$ w/ absolute convergence if $|\nu(E)| < \infty$.

Props like monotonicity and sub-additivity are easily seen not to hold, but cont. from above and below hold (w/ essentially the same pf).

Def 2. A set $E \in \mathcal{M}$ is <u>negative/null/positive</u> for ν if all meas. subsets $F \subseteq E$ satisfy $\nu(F) \leq 0 / = 0 / \geq 0$.

<u>Hahn Decomposition Thm.</u> Let ν be a signed measure on $(X, \mathcal{M})$. Then, $\exists$ P positive, N negative s.t. $P \cap N = \emptyset$ and $X = P \cup N$. If P', N' are other such sets, then $P \Delta P'$, $N \Delta N'$ are null.

<u>Pf.</u> Assume ν does not take value $+\infty$. Let $\mathcal{P} = \{E \in \mathcal{M} : E \text{ positive}\}$. Note that $\emptyset \in \mathcal{P}$. Let $\alpha = \sup\limits_{\mathcal{P}} \nu$. <u>WLOG</u> $\alpha > 0$. (If $\alpha = 0$, then $-\nu$ is positive measure and we take $N = X$, $P = \emptyset$.)

Let $P_n \in \mathcal{P}$ be seq. s.t. $\nu(P_n) \nearrow \alpha$ and let $P = \bigcup\limits_{n=1}^{\infty} P_n$.

Note that for a positive set A, we do have monotonicity and subadditivity $\Rightarrow \nu(P) \geq \lim_{k\to\infty} \nu(P_k) = \alpha$.

Also, it is easy to see that countable unions of positive sets are positive $\Rightarrow$ P is positive $\Rightarrow \nu(P) = \alpha$. ($\Rightarrow \alpha < \infty$).

Let $N = X \setminus P \Rightarrow X = N \cup P$, $N \cap P = \emptyset$.

<u>Claim.</u> N is negative.

<u>Pf.</u> Suppose not. Then $\exists A \subseteq N$ s.t. $\nu(A) > 0$. But A cannot be positive because otherwise $P \cup A$ would be positive and $\nu(P \cup A) = \alpha + \nu(A) > \nu(A)$

Thus, $\exists B \subseteq A$ s.t. $\nu(B) < 0 \Rightarrow$ $\nu(A \setminus B) > \nu(A)$.

<u>Construction</u>: Let $n_1 \in \mathbb{Z}_+$ be minimal s.t. $\exists A_1 \subseteq N$ w/ $\nu(A_1) \geq \frac{1}{n_1}$. By using the above observation, we

inductively construct $n_j \in \mathbb{Z}_+$ minimal and $A_j \subseteq A_{j-1}$ s.t. $\nu(A_j) \geq \nu(A_{j-1}) + \frac{1}{n_j}$.

Set $A = \bigcap_{j\geq 1}^{\infty} A_j$. Since $\nu(A_1)$ is finite by the assumption that ν does not assume the value $+\infty$, cont. from above $\Rightarrow$

$$\infty > \nu(A) = \lim_{j\to\infty} \nu(A_j) \geq \lim_{j\to\infty} \sum_{i=1}^{j} \frac{1}{n_i} = \sum_{i=1}^{\infty} \frac{1}{n_i}$$

$\Rightarrow n_i \to \infty$. But, again, A is not positive $\Rightarrow \exists B \subseteq A$ s.t. $\nu(B) \geq \nu(A) + \frac{1}{n}$ for some n. This contradicts the construction above, since $n_i \to \infty \Rightarrow$ $\exists i$ s.t. $n_i > n$ but n_i was minimal s.t. $\nu(A_i) \geq \nu(A_{i-1}) + \frac{1}{n_i}$.

Since $\nu(A) \geq \nu(A_{i-1}) \Rightarrow$

$$\nu(A_{i-1}) + \frac{1}{n} < \nu(A) + \frac{1}{n} < \nu(B) \leq \nu(A_{i-1}) + \frac{1}{n_i}$$

$\Rightarrow n < n_i$. $\lightning$. The uniqueness of decomp. PUN mod nullsets is easy. $\boxtimes$

Def. 2 Given signed measures ν_1, ν_2 on $(X, \mathcal{M})$, say ν_1, ν_2 mutually singular, $\nu_1 \perp \nu_2$, if $\exists$ decomp. $X = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, s.t. E_1 null for ν_2 and vice versa.

Jordan Decomp. Thm. If ν is signed measure $\exists$ unique positive measures ν^+, ν^- s.t. $\nu^+ \perp \nu^-$ and $\nu = \nu^+ - \nu^-$.

Pf. Use Hahn Decomp. $X = P \cup N$ and let $\nu^+ = \nu|_P$, $\nu^- = -\nu|_N$. ∎

Moreover:

- $|\nu| = \nu^+ + \nu^-$ is positive measure.
  pos. variation, neg. variation.
- Define for $f \in L^+, L^1(\nu^+) \cap L^1(\nu^-) = L^1(\nu)$
  $$\int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-.$$

Absolute continuity.

Def. If ν signed, μ positive measures on $(X, \mathcal{M})$, then $\nu << \mu$ (ν absolutely continuous w.r.t. μ) provided

$$\mu(E) = 0 \Rightarrow \nu(E) = 0.$$

Rem. $\nu << \mu \Leftrightarrow |\nu| << \mu \Leftrightarrow \nu^+, \nu^- << \mu$. (HW)

Thm 1. Assume ν finite, signed and μ positive measures. Then

$$\nu << \mu \Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \mu(E) < \delta \Rightarrow |\nu|(E) < \varepsilon.$$

Pf. Suffices to consider the special case where ν also positive. "$\Leftarrow$" is obvious. We prove "$\Rightarrow$".

Suppose not. Then $\exists\, \varepsilon>0$ and $E_k \in \mathcal{M}$ s.t. $\mu(E_k) \le 2^{-k}$ but $\nu(E_k) \ge \varepsilon$.

Let $F_n = \bigcup_{k=n}^{\infty} E_k$ (decreasing), $F = \bigcap_{n=1}^{\infty} F_n$

$\Rightarrow$

$$\begin{cases} \nu(F_n) \ge \nu(E_n) \ge \varepsilon \\ \mu(F_n) \le \sum_{k=n}^{\infty} \mu(E_k) \le 2^{-n+1}. \end{cases}$$

Since $F \subseteq F_n$, $\forall n$, $\Rightarrow \mu(F) \le 2^{-n+1} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow \mu(F) = 0$.

Since ν finite, $\overbrace{\text{cont. from above}}^{F_n \downarrow F} \Rightarrow$

$\nu(F) = \lim_{n\to} \nu(F_n) \ge \varepsilon$.

$\Rightarrow \nu \not\ll \mu$. Thus, $\nu \ll \mu \Rightarrow$ ε-δ statement.

$\boxtimes$